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Discrete Applied Mathematics 127 (2003) 399–414

DISCRETE APPLIED MATHEMATICS

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The power of small coalitions in graphs ☆

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Received 3 May 2000; received in revised form 11 October 2001; accepted 12 November 2001

Abstract

This paper considers the question of the influence of a coalition of vertices, seeking to gain control (or majority) in local neighborhoods in a general graph. Say that a vertex v is *controlled* by the coalition M if the majority of its neighbors are from M. We ask how many vertices (as a function of $|M|$) can M control in this fashion. Upper and lower bounds are provided for this problem, as well as for cases where the majority is computed over larger neighborhoods (either neighborhoods of some fixed radius $r \geqslant 1$, or all neighborhoods of radii up to r). In particular, we look also at the case where the coalition must control all vertices (including or excluding its own), and derive bounds for its size.

1. Introduction

This paper considers the question of the influence of a coalition of vertices, seeking to gain control in local neighborhoods in a general graph. This problem is motivated by fault tolerance and recovery applications in distributed computing, where decisions are taken after a voting process using a majority rule, cf. [7].

The basic notion needed towards formally defining our question is the following.

☆ This work was partly supported by a grant from AFIRST and the Israel Ministry of Science and Arts (in the framework of the French–Israeli cooperation). Parts of this paper have appeared in preliminary form in [2,3].

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1 Supported in part by a grant from the Israel Science Foundation.

PII: S0166-218X(02)00241-X

Definition 1.1. A vertex v in a network $G(V,E)$ is said to be *controlled* by the vertex set M if the majority of its neighbors are in M.

Since our focus is on obtaining asymptotic results, there are a number of slightly different definitions for the terms "neigborhood" and "majority" that we can use in the above definition, without affecting the results. For concreteness, let us define the *neighborhood* of v as including the vertex v itself and all vertices adjacent to it, and *majority* as a strict one.

Taking the "adversarial" point of view, we formulate the following initial question:

(Q1) How many vertices (as a function of $|M|$) can a set M control?

It turns out that as far as extremal behavior is concerned, question (Q1) is easy to answer: control of virtually all vertex neighborhoods can be achieved by extremely small coalitions. Specifically, in a full bipartite graph $\tilde{G}$ with one of the two bipartitions being a 2-vertex set, $\tilde{M}=\{a,b\}$, $\tilde{M}$ can gain control over the majority of the neighbors for every other vertex in $V \setminus \tilde{M}$.

The curious phenomenon illustrated by the above example may be viewed as an outcome of the limited scope of our majority voting. Indeed, one may hope to strengthen the quality of the voting by querying vertices to larger distances. Let $\Gamma_r(v)$ denote the r-neighborhood of v, i.e., the set of vertices at distance r or less from v (including v itself). We next pose a variant of the above question, in which neighborhoods are replaced by r-neighborhoods for some fixed r:

Definition 1.2. A vertex v in a network $G(V,E)$ is said to be r-*controlled* by the vertex set M if the majority of the vertices in $\Gamma_r(v)$ are in M.

(Q2) How many vertices can a set M r-control?

It turns out that an extremal behavior similar to that of the above example $(\tilde{M},\tilde{G})$ may occur for r-control as well, on certain graphs. More precisely, we shall present examples for every integer $r \geqslant 1$, in which a set M of size $r+1$ can r-control as many as $(n-r-1)/r$ vertices in an n-vertex graph G.

A more interesting picture emerges if we strengthen our voting policy, and examine *all* i-neighborhoods for a *range* of values of i.

Definition 1.3. A vertex v in a network $G(V,E)$ is said to be $[1,r]$-*controlled* by the vertex set M if for every $1 \leqslant i \leqslant r$, the majority of the vertices in $\Gamma_i(v)$ are in M.

(Q3) How many vertices can a set M $[1,r]$-control?

Our results imply that the answer to this last question is[2] $O(|M|^{1+1/\lfloor \log_2 r \rfloor})$ and that this result is tight, in the sense that there exist (infinitely many) graphs and sets M that achieve this influence.

A special case of the above problems was raised and studied in [6]. It is based on the following notion.

[2] Using the standard big-oh, big-omega and big-theta notation, cf. [4].

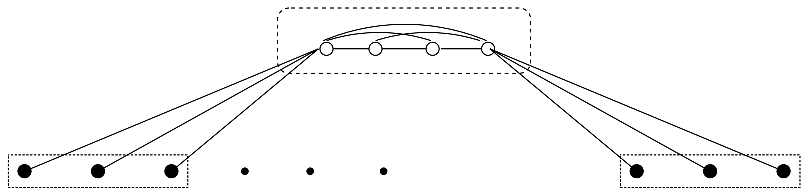

Fig. 1. A graph $G^{\mathbf{A}}$ with a monopoly M of size $O(\sqrt{n})$. (In all our figures, white circles represent vertices of the coalition, and black circles represent vertices controlled by the coalition.)

Definition 1.4. Call the set M an *r-monopoly* (respectively *$[1,r]$-monopoly*) if it r-controls (respectively $[1,r]$-*controls*) every vertex in the graph.

The question addressed in [6] was:

What can be said about the size of monopolies in the graph?

Tight answers to this question was provided in [6], by relating it to some natural packing and covering problems in graphs. Specifically, the following results were established in [6]. Let $g_r(n)$ (respectively, $g_{[1,r]}(n)$) denote the size of the smallest r-monopoly (resp., $[1,r]$-monopoly) in any n-vertex graph.

Proposition 1.5 (Linial et al. [6]). $g_1(n) = \Theta(\sqrt{n})$.

A graph $G^{\mathbf{A}}$ with a 1-monopoly of size $O(\sqrt{n})$ as in Proposition 1.5 is depicted in Fig. 1. The graph consists of a coalition M of $2\sqrt{n}$ vertices, u_i, w_i for $1 \leqslant i \leqslant \sqrt{n}$, connected by a clique. The rest of the vertices are partitioned into $\sqrt{n}$ groups of $\sqrt{n}-2$ vertices each, where the vertices of the ith group are attached to u_i and w_i.

As for r-monopolies, it is shown in [6] that $g_r(n) = O(n^{2/3})$ for any fixed $r \geqslant 1$. However, the question of tight bounds for r-monopolies was left open. In this paper we improve this bound on the size of r-monopolies. Specifically, we show the following.

Theorem 1.6. *For every even* $r \geqslant 2$, $g_r(n) = \Theta(n^{3/5})$.

For odd r values we show the following.

Theorem 1.7. *For every odd* $r \geqslant 3$, $g_r(n) = \Omega(n^{6/11})$.

Theorem 1.8. *For every* $r = 3k$ *for fixed odd* k, $g_r(n) = O(n^{4/7})$.

Finally, the case of $[1,r]$-monopolies was also given tight bounds in [6].

Proposition 1.9 (Linial et al. [6]). $g_{[1,r]}(n) = \Theta(n^{1-1/(\lfloor \log_2 r \rfloor + 2)})$.

While the surprising power of small coalitions is clearly demonstrated in the results of [6], there are settings in which controlling coalitions can be even smaller. In particular, in a context where we think of the coalition seeking control as a set of faulty (possibly malicious) processors, it may as well be assumed that the coalition M is only interested

max. radius r	$[1,r]$-monopoly	self-ignoring $[1,r]$-monopoly
1	$\Theta(n^{1/2})$	$\Theta(1)$
2,3	$\Theta(n^{2/3})$	$\Theta(n^{1/2})$
4,5,6,7	$\Theta(n^{3/4})$	$\Theta(n^{2/3})$
2^{t-1} to $2^t - 1$	$\Theta(n^{1-1/(t+1)})$	$\Theta(n^{1-1/t})$

Fig. 2. Size comparison of $[1,r]$-monopolies vs. self-ignoring $[1,r]$-monopolies.

in gaining control over the neighborhoods of *other* vertices, belonging to $V \setminus M$. This is because the vertices in the coalition are not obligated by the rules of the "voting game" anyhow, so the "adversary" needs not "waste its powers" (so to speak) on controlling them. Such a coalition can therefore be considerably smaller. For instance, the set $\tilde{M} = \{a,b\}$ in the above example controls every vertex in $V \setminus M$, in sharp contrast with Proposition 1.5.

More generally, we can define the following notion.

Definition 1.10. A *self-ignoring r-monopoly M* is a set that r-controls every vertex in $V \setminus M$ (and similarly for a self-ignoring $[1,r]$-monopoly).

We can now repeat the questions of [6] for self-ignoring monopolies, letting $g_r^{\mathrm{SI}}(n)$ (respectively, $g_{[1,r]}^{\mathrm{SI}}(n)$) denote the size of the smallest self-ignoring r-monopoly (resp., $[1,r]$-monopoly) in any n-vertex graph. It turns out that the results have a rather similar structure, except "shifted" downwards. In particular, we prove the following.

Theorem 1.11. *For every fixed* $r \geqslant 2$, $g_r^{\mathrm{SI}}(n) = \Theta(n^{1/2})$.

(For $r = 1$, the example $(\tilde{M}, \tilde{G})$ given earlier prevents such lower bound.)

Turning to self-ignoring $[1,r]$-monopolies, tight bounds follow from our bounds on the extent of control possible for vertex sets.

Theorem 1.12. *For every fixed integer* $r \geqslant 1$, $g_{[1,r]}^{\mathrm{SI}} = \Omega(n^{1-1/(\lfloor \log_2 r \rfloor + 1)})$.

Note again the slight difference in the exponent between the bounds for $[1,r]$-monopolies and self-ignoring $[1,r]$-monopolies (see Fig. 2).

A bound of the form $\Omega(n^{1-1/(\lfloor \log_2 r \rfloor + 1)})$ can be derived for both cases using (slightly different variants of) the integral packing technique developed in [6]. This bound is tight for the self-ignoring case, but for the case of a full monopoly, proving Proposition 1.9 is done via a different technique for constructing fractional packings.

Another related concept is that of signed domination (cf. [5]). In particular, the minimum cardinality of a 1-monopoly is directly related to the signed domination number of the graph.

In a related research area, certain *dynamic* variants of majority voting problems were studied in the literature, in the context of discrete time dynamical systems. These variants concentrated on a setting in which the nodes of the system operate in discrete time steps, and at each step, each node computes the majority in its neighborhood, and adapts the resulting value as its own. The typical problems studied in this setting involve the behavior of the resulting sequence of global states (represented as a vector $\bar{x}^t = (x_1^t, \ldots, x_n^t)$, where x_i^t represents the value at node v_i after time step t). For more details and additional references on the dynamic versions of the problem, as well as a review of the area and its applications in the fields of fault tolerance and distributed computing, see [7].

2. Small r-controlling coalitions

For r-control, a very small set M can r-control a very large set of vertices. For $r=1$ this is demonstrated by the example $(\tilde{M}, \tilde{G})$ given in the introduction, where a set $\tilde{M}$ of size 2 achieves 1-control over the remaining $n-2$ vertices. This example can be generalized to show the following.

Theorem 2.1. *For any integer r there exists a family of n-vertex graphs G_n and sets M_n, such that M_n r-controls a subset X_n of $V \setminus M_n$, and $|M_n| = r+1$, $|X_n| = (n-r-1)/r$.*

Proof. Given r and p, let $n = rp + (r+1)$. Construct $G^{\mathbf{C}}_{r,p}$ as follows. The graph is *leveled*, namely, the vertices are arranged into $r+1$ levels, numbered 1 through $r+1$, with edges connecting only vertices in adjacent levels ℓ, $\ell+1$. Each level $2 \leqslant \ell \leqslant r+1$ contains p vertices, $v_1^\ell, \ldots, v_p^\ell$, and level 1 contains $r+1$ vertices. Let X denote the set of vertices on level $r+1$, and let M denote the set of vertices on level 1. When p is very large with respect to r, X contains roughly a $1/r$ fraction of the vertices of the graph, yet the edge connections defined next will guarantee that M has the majority in any r-neighborhood around the vertices of X.

The edges connecting two consecutive levels $\ell-1$ and ℓ are defined as follows. The vertices of level 1 (M) are connected by a complete bipartite graph (crossbar) to the vertices of level 2. From level 2 and on, the vertices of the different levels form chains of length r. Namely, for $2 \leqslant \ell \leqslant r$, each vertex v_i^ℓ of level ℓ is connected to vertex $v_i^{\ell+1}$ of level $\ell+1$. Fig. 3 depicts an example graph $G^{\mathbf{C}}_{r,p}$ for $r=3$ and some p. It is straightforward to verify that the vertices of M r-control those of X. $\square$

3. r-monopolies

3.1. Lower bound for 2-monopolies

Given a graph $G=(V,E)$, a vertex $x \in V$, and a set $S \subset V$, we denote by $\deg_G(x,S)$ the number of neighbors of x in G belonging to S, namely, $|\Gamma_1(x) \cap S|$. (We omit the parameter S when it is the entire vertex set of G; we omit the subscript G when it is clear from the context.) $D_G(x,y)$ denotes the distance in G between x and y. Given a subset S of V, define the distance from x to S in G as $D_G(x,S) = \min_{y \in S}(D_G(x,y))$.

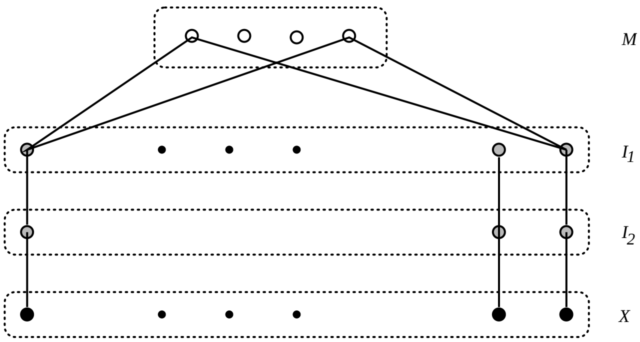

Fig. 3. The graph $G^{\mathrm{C}}_{r,p}$ for $r = 3$ and some p, with a set M of size 4 controlling the majority of 3-neighborhoods of the vertices of a set X of size $p = (n-4)/3$. (Uncontrolled vertices are represented by gray circles.)

In this subsection we concentrate on the case of 2-monopolies. We will refer to the pair (G,M) as a "2-monopoly" whenever G is a graph (V,E) and $M \subseteq V$ is 2-monopoly for G.

For every $i \geqslant 1$, let S_i denote the set of vertices at distance i from M, that is, $S_i = \{v \in V \mid D_G(v,M) = i\}$, and let $s_i = |S_i|$. Also let $m = |M|$. Note that if (G,M) is a 2-monopoly then $V = M \cup S_1 \cup S_2$ and $n = m + s_1 + s_2$.

The *influence* of a subset $S \subset V$ on a vertex v in the graph G is defined by

$$\mathscr{I}_G(S,v) = \Gamma_2(v) \cap S = |\{w \in S \mid D_G(v,w) \leqslant 2\}|.$$

(We omit the subscript G whenever clear from the context.) More generally, the influence of a set S on a set S' is $\mathscr{I}_G(S,S') = \sum_{x \in S'} \mathscr{I}_G(S,x)$.

Using the $\mathscr{I}$ notation, a vertex v is 2-controlled by M if $\mathscr{I}(M,v) > \mathscr{I}(S_1 \cup S_2, v)$ and a subset S of V is 2-controlled if every vertex $x \in S$ is.

In what follows we often make use of the following well-known fact.

Fact 1. *For fixed $a > 1$ and $s > 0$, the function $\sum_{i=1}^{k} x_i^a$ attains its minimum over the range constrained by $\sum_{i=1}^{k} x_i = s$ and $x_1, \ldots, x_k > 0$ at the point $x_1 = \cdots = x_k = s/k$.*

We will also make frequent use (sometimes without mention) of the following three immediate properties of $\mathscr{I}$.

Proposition 3.1. (1) *$\mathscr{I}$ is a symmetric function, i.e., $\mathscr{I}(S,S') = \mathscr{I}(S',S)$ for every $S,S' \subseteq V$.*

(2) *$\mathscr{I}(S,S) \leqslant |S|^2$ for every $S \subseteq V$, and*

(3) *$\mathscr{I}$ is monotone nonincreasing in the size of E, namely, if $G' = (V,E')$ for $E' \subseteq E$, then $\mathscr{I}_{G'}(S,S') \leqslant \mathscr{I}_G(S,S')$ for every $S,S' \subseteq V$.*

Lemma 3.2. *If (G,M) is a 2-monopoly, then $\mathscr{I}(S_1 \cup S_2, M) \leqslant m^2$ and $\mathscr{I}(S_1 \cup S_2, S_1 \cup S_2) \leqslant m^2$.*

Proof. As all the nodes are 2-controlled, we have for every $v \in V$,

$$\mathscr{I}(S_1 \cup S_2, v) \leqslant \mathscr{I}(M, v). \tag{1}$$

By summing (1) on all $v \in M$ we get

$$\mathscr{I}(S_1 \cup S_2, M) \leqslant \mathscr{I}(M, M) \leqslant m^2 \tag{2}$$

proving the first claim. On the other hand, by summing (1) on all $v \in S_1 \cup S_2$ we get

$$\mathscr{I}(S_1 \cup S_2, S_1 \cup S_2) \leqslant \mathscr{I}(M, S_1 \cup S_2) = \mathscr{I}(S_1 \cup S_2, M) \tag{3}$$

and combining (2) and (3), the lemma follows. □

Our main two lemmas bound the size of S_1 and S_2 with respect to M.

Lemma 3.3. *If (G, M) is a 2-monopoly, then $s_1 \leqslant m^{3/2}$.*

Proof. For every vertex $y \in S_1$, we assign a unique neighbor $p(y)$ in M as its *parent*. For every $x \in M$, let $\alpha(x)$ denote the number of children assigned to it. Note that $s_1 = \sum_{x \in M} \alpha(x)$. Also note that for every vertex $y \in S_1$, $\alpha(p(y)) \leqslant \mathscr{I}(y, S_1) \leqslant \mathscr{I}(y, M)$. Therefore,

$$\sum_{x \in M} \alpha^2(x) = \sum_{y \in S_1} \alpha(p(y)) \leqslant \sum_{y \in S_1} \mathscr{I}(y, M) = \mathscr{I}(S_1, M) \leqslant m^2.$$

By Fact 1, this sum is minimum when $\alpha(x) = s_1/m$ for every $x \in M$, as $s_1 = \sum_{x \in M} \alpha(x)$. Hence $m(s_1/m)^2 \leqslant m^2$, yielding the claim. □

Consider a graph G and a set of vertices M with S_1 and S_2 defined as above. Let E_1 denote the set of edges connecting M and S_1. The following lemma holds even if M is not a 2-monopoly.

Lemma 3.4. *If S_2 is 2-controlled by M, then $|E_1| \geqslant s_2$.*

Proof. Construct a bipartite graph $B = (S_2, E_1, E')$ by defining the edge set E' as follows. For $z \in S_2$, $y \in S_1$ and $x \in M$ such that $(x, y) \in E_1$, we connect z to (x, y) in B if z is adjacent to y in G.

We prove the lemma by showing that B admits a matching saturating S_2. This is proved by relying on Hall's Lemma (cf. [1]). For $U \subseteq S_2$, let $K(U)$ denote the set of vertices in B connected to some $u \in U$. By Hall's Lemma, B admits a matching touching every vertex of S_2 if and only if $|K(U)| \geqslant |U|$ for every $U \subseteq S_2$.

This property is proved by contradiction. Suppose that this property does not hold, namely, there exists a "deficit" set U, such that $|K(U)| < |U|$. Let U_0 be a minimum size deficit set. Note that U_0 is not a singleton (since no singletons in S_2 are in deficit). Let $F = K(U_0)$. Pick an arbitrary node $z_0 \in U_0$, and let $Z = U_0 \setminus \{z_0\}$. The bipartite graph B' induced by Z and $K(Z)$ obeys the condition of Hall's Lemma, hence it admits a matching saturating the vertices of Z. Moreover, note that

$$|Z| + 1 = |U_0| > |K(U_0)| = |F| \geqslant |K(Z)| \geqslant |Z|,$$

so $|F| = |Z|$, and hence B' admits a perfect matching $\mathscr{M}$ between Z and F. Now let $Q = K(\{z_0\}) \subseteq F$, and let H be the set of nodes in Z matched by $\mathscr{M}$ with the edges of Q. Map each edge $q \in Q$ to the M vertex it touches. This maps Q *onto* $\Gamma_2(z_0, G) \cap M$. Hence,

$$|\Gamma_2(z_0, G) \cap M| \leqslant |Q| = |H| \leqslant |\Gamma_2(z_0, G) \cap S_2| - 1$$

(counting z_0 itself, and possibly more S_2 vertices currently not in U_0). This implies that z_0 is not 2-controlled, leading to contradiction.

Consequently, B admits a matching saturating S_2, and hence $s_2 \leqslant |E_1|$. □

Hereafter, let $\delta(y) = \deg_G(y, M)$ for each node $y \in S_1$.

Lemma 3.5. *If (G, M) is a 2-monopoly, then $s_2 \leqslant O(m^{5/3})$.*

Proof. Consider a 2-monopoly (G, M). Let d_0 be an integer to be fixed later. Let A_1 (resp. B_1) be the set of vertices $y \in S_1$ with $\delta(y) < d_0$ (resp. $\delta(y) \geqslant d_0$). Let B_2 be the set of nodes in S_2 adjacent to some node in B_1, and $A_2 = S_2 \setminus B_2$. Let $a_2 = |A_2|$ and $b_2 = |B_2|$.

First, we note that $m^2 \geqslant \mathscr{I}(B_2, M) \geqslant b_2 d_0$, leading to

$$b_2 \leqslant \frac{m^2}{d_0}. \tag{4}$$

Secondly, let t be the number of edges between M and A_1. Note that a vertex $z \in A_2$ has all its S_1 neighbors in A_1, and consequently, A_2 is 2-controlled by M in the subgraph of G induced by $M \cup A_1 \cup A_2$. Therefore $t \geqslant a_2$ by Lemma 3.4. These edges form paths of length 2 from A_1 to A_1 via M. (Note that these paths may include edge repetitions.) The number of such paths is $K = \sum_{x \in M} \deg(x, A_1)^2$. By Fact 1, this number is minimum when the degrees $\deg(x, A_1)$ are equal, in which case $\deg(x, A_1)) = t/m \geqslant a_2/m$ for any $x \in M$. Hence,

$$K \geqslant \left(\frac{a_2}{m}\right)^2 m = \frac{a_2^2}{m}.$$

As we want to compute the influence $\mathscr{I}(A_1, A_1)$, we have to take into account the fact that a vertex in A_1 may influence another vertex in A_1 via more than one 2-path. But as for any $y \in A_1$ we have $\deg(y, M) \leqslant d_0$, the number of different 2-paths via M that can contribute the (same) influence of some $y' \in A_1$ on y is at most d_0. Consequently,

$$m^2 \geqslant \mathscr{I}(S_1, S_1) \geqslant \mathscr{I}(A_1, A_1) \geqslant \frac{K}{d_0} \geqslant \frac{a_2^2}{m d_0}$$

and we get

$$a_2 \leqslant \sqrt{d_0} m^{3/2}. \tag{5}$$

The upper bound on $s_2 = a_2 + b_2$ is minimized for $d_0 = (4m)^{1/3}$. Substituting this value in inequalities (4) and (5) yields $a_2 \leqslant 2^{1/3} m^{5/3}$, $b_2 \leqslant 4^{-1/3} m^{5/3}$ and $s_2 \leqslant (2^{1/3} + 4^{-1/3}) m^{5/3}$. □

Since $n = s_1 + s_2 + m$, by Lemmas 3.3 and 3.5 we have the following.

Theorem 3.6. $g_2(n) = \Omega(n^{3/5})$.

3.2. Lower bound for $2k$-monopolies

The generalization to $2k$-monopolies is straightforward, and yields the first direction of Theorem 1.6.

Theorem 3.7. *For every even $r \geqslant 2$, $g_r(n) = \Omega(n^{3/5})$.*

Proof. Let (G,M) be a $2k$-monopoly. Define G^k as the graph on vertex set V with an edge between x and y if and only if $D_G(x,y) \leqslant k$. Then clearly (G,M) is a $2k$-monopoly only if (G^k,M) is a 2-monopoly. □

3.3. Lower bound for $2k+1$-monopolies

We now prove Theorem 1.7.

Proof of Theorem 1.7. Let us first prove the claim for $r = 3$. We use the terminology defined at the beginning of Section 3.1. For every vertex $z \in S_3$ we associate a unique parent $p(z)$ in S_2, and for every vertex $y \in S_2$ we associate a unique parent $p(y)$ in S_1. This induces for each vertex $y \in S_2$ a set $C(y)$ of *children* in S_3, namely, $C(y) = \{z \in S_3 \mid p(z) = y\}$, and similarly, for each vertex $x \in S_1$ we have $C(x) = \{y \in S_2 \mid p(y) = x\}$. Denote the number of children of every vertex $y \in S_2$ by $d(y) = |C(y)|$. Finally, for a vertex $x \in S_1$, let $a(x) = |C(x)|$ and $b(x) = \sum_{y \in C(x)} d(y)$.

Construct a bipartite graph $B = (S_3, M, E')$ by defining the edge set E' as follows. For $z \in S_3$ and $v \in M$, E' contains an edge (v,z) if z is at distance 3 from v in G. Let q_z (respectively, t_v) denote the degree of each $z \in S_3$ (resp., $v \in M$) in B. Note that $\sum_{z \in S_3} q_z = \sum_{v \in M} t_v \leqslant m^2$. Also note that every $z \in S_3$ is influenced by at least $d(p(z)) + a(p(p(z)))$ vertices outside M, hence to be controlled by M, it must satisfy $q_z \geqslant d(p(z)) + a(p(p(z)))$. It follows that

$$\begin{aligned} m^2 &\geqslant \sum_{z \in S_3} (d(p(z)) + a(p(p(z)))) = \sum_{y \in S_2} (d^2(y) + d(y)a(p(y))) \\ &= \sum_{x \in S_1} \left(\sum_{y \in C(x)} d^2(y) + a(x) \sum_{y \in C(x)} d(y) \right). \end{aligned}$$

By the Cauchy–Schwarz inequality:

$$\begin{aligned} m^2 &\geqslant \sum_{x \in S_1} \left(\frac{1}{a(x)} \left(\sum_{y \in C(x)} d(y) \right)^2 + a(x) \sum_{y \in C(x)} d(y) \right) \\ &= \sum_{x \in S_1} \left(\frac{b^2(x)}{a(x)} + a(x)b(x) \right) \geqslant \sum_{x \in S_1} b^{3/2}(x). \end{aligned}$$

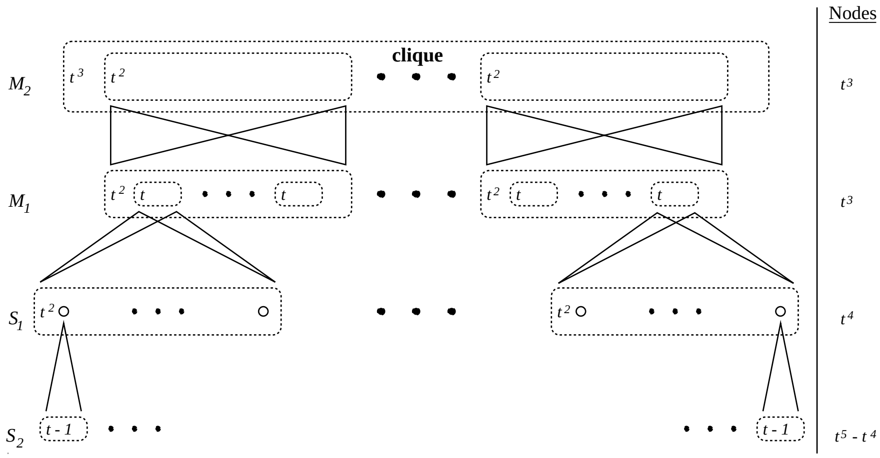

Fig. 4. A graph $G^{\mathbf{D}}$ with a 2-monopoly M of size $O(n^{3/5})$.

By Fact 1, we get

$$m^2 \geqslant |S_1| \left(\frac{\sum_{x \in S_1} b(x)}{|S_1|} \right)^{3/2} = \frac{|S_3|^{3/2}}{|S_1|^{1/2}}.$$

As $|S_1| \leqslant m^{3/2}$ (by an argument similar to the proof of Lemma 3.3), we get that $m^2 \geqslant |S_3|^{3/2}/m^{3/4}$, hence $|S_3|^{3/2} \leqslant m^{11/4}$, or $|S_3| \leqslant m^{11/6}$. As also $|S_2| \leqslant m^{5/3}$ (by an argument similar to the proof of Lemma 3.5), and since $n = s_1 + s_2 + s_3 + m$, we get the required bound.

The bound is generalized to $(2k+1)$-monopolies for $k \geqslant 2$ in a straightforward manner. □

3.4. Upper bounds

Next we prove the other direction of Theorem 1.6.

Theorem 3.8. *For every fixed $r \geqslant 2$, $g_r(n) = O(n^{3/5})$.*

Proof. To prove the theorem we construct a graph $G^{\mathbf{D}}$ with a 2-monopoly associated to a parameter t with $m = \Theta(t^3)$, $s_1 = \Theta(t^4)$, and $n = \Theta(t^5)$, that is, $n = \Theta(m^{5/3})$. The nodes $x \in S_1$ also satisfy $\delta(x) = O(t) = O(m^{1/3})$. Clearly, these parameters ensure the lower bound. An outline of the construction is given in Fig. 4.

- $V = M_2 \cup M_1 \cup S_1 \cup S_2$.
- M_2 is a clique of size t^3, composed of t sets of t^2 vertices.
- M_1 is a independent set of size t^3 composed of t sets of t^2 vertices. The ith set in M_1 is connected to the ith set in M_2 by a complete bipartite graph. Each set of size t^2 is decomposed into subsets of size t.

- S_1 is a independent set of size t^4, composed of t^2 sets of size t^2. The ith set in S_1 (of size t^2) is connected to the ith subset in M_1 (of size t) by a complete bipartite graph.
- S_2 is a independent set of size $t^5 - t^4$, composed of t^4 sets of size $t-1$. The nodes of the ith set in S_2 are connected to the ith node of S_1.

In order to show that the construction gives a 2-monopoly achieving the lower bound it is enough to count the influences for the four types of vertices (in M_2, M_1, S_1, S_2). The following table summarizes the counts.

	$x \in M_2$	$x \in M_1$	$x \in S_1$	$x \in S_2$
$\mathcal{I}(x, M)$	$2t^3$	$t^3 + t^2$	$t^2 + t$	t
$\mathcal{I}(x, S)$	t^3	t^3	$t^2 + t - 1$	t

The construction technique can easily be extended to the case of r-monopolies for $r > 2$; this extension is left to the reader. □

Proof of Theorem 1.8. To prove the theorem for $r = 3$, we construct a graph G^{E} with a 3-monopoly M associated to a parameter t with $m = \Theta(t^4)$, $s_1 = \Theta(t^5)$, and $n = \Theta(t^7)$, that is, $n = \Theta(m^{4/7})$. Clearly, these parameters ensure the lower bound. An outline of the construction is given in Fig. 5. The set M consists of the top three levels of the graph. The details of the construction are similar to those of the previous construction from Theorem 3.8, except that there are six levels overall, rather than only four for $r = 2$.

The natural extension to any $r \geqslant 9$ is omitted. □

4. Self-ignoring r-monopolies

Finally, we turn to self-ignoring r-monopolies. As mentioned in the introduction, there exist graphs with self-ignoring 1-monopolies of size 2. We now derive a lower bound on the size of self-ignoring r-monopolies for $r \geqslant 2$, thus proving the first direction of Theorem 1.11.

Theorem 4.1. *For every fixed* $r \geqslant 2$, $g_r^{\mathrm{SI}}(n) = \Omega(n^{1/2})$.

Proof. Consider a graph G and a self-ignoring r-monopoly M, for $r \geqslant 2$. Let $S_i = \bigcup_{v \in M} \Gamma_i(v)$ for every $1 \leqslant i \leqslant r$. Let $M_1 = S_1 \setminus M$ and $M_i = S_i \setminus S_{i-1}$. Any vertex of V is at distance at most r from M, so $V \setminus M = \bigcup_{1 \leqslant i \leqslant r} M_i$. For every $1 \leqslant i \leqslant r-1$, let $C(M, M_i)$ denote the set of pairs (v, w) with $v \in M$ and $w \in M_i \cap \Gamma_i(v)$.

Fact 2. (a) *For every* $1 \leqslant i \leqslant r-1$, $|C(M, M_i)| \leqslant |M|^2$.
(b) *For every* $1 \leqslant i \leqslant r-1$, $|M_i| < |M|^2$.

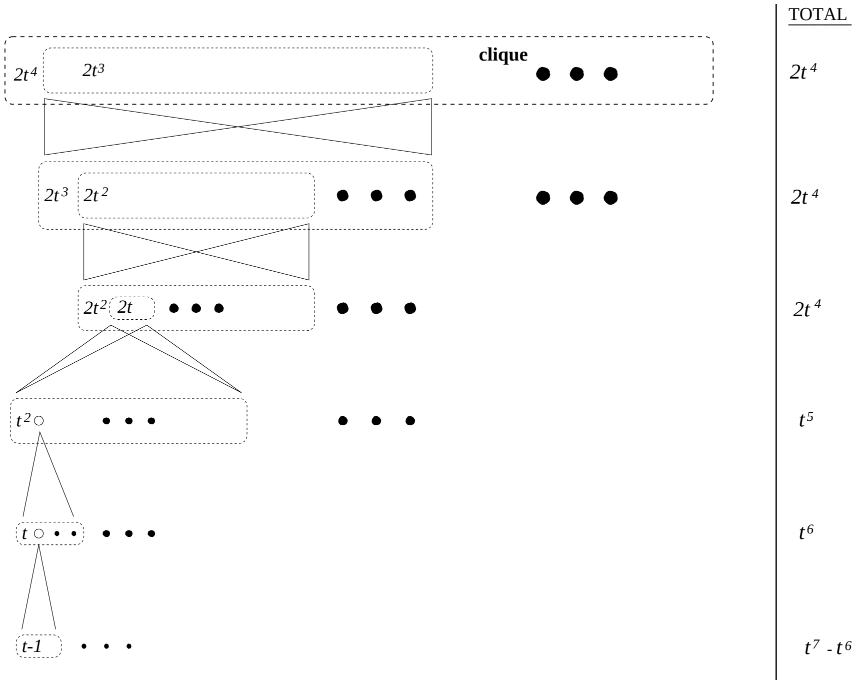

Fig. 5. A graph $G^{\mathbf{E}}$ with a 3-monopoly M of size $\mathrm{O}(n^{4/7})$.

Proof. Let us first prove that

$$|\Gamma_i(v) \cap M_i| < |M| \quad \text{for every } 1 \leqslant i \leqslant r-1 \quad \text{and} \quad v \in M. \tag{6}$$

Clearly, if v has no neighbors in M_1, then $\Gamma_i(v) \cap M_i = \emptyset$, and (6) follows. So now suppose v has such neighbors, and let w be a vertex in M_1 adjacent to v. Then $\Gamma_r(w) \supseteq \Gamma_i(v) \cap M_i$ so that $|\Gamma_i(v) \cap M_i| \leqslant |\Gamma_r(w) \cap M| < |M|$, and again (6) follows.

Claim (a) now follows by noting that $C(M,M_i) = \bigcup_{v \in M} \{(v,w) \mid w \in \Gamma_i(v) \cap M_i\}$, and therefore $|C(M,M_i)| \leqslant \sum_{v \in M} |\Gamma_i(v) \cap M_i|$, which by (6) is strictly smaller than $|M|^2$.

Finally, claim (b) follows from claim (a) as $|M_i| \leqslant |C(M,M_i)|$. $\square$

Now let us prove that $|M_r| < |M|^2$. We will use the following relations. For every vertex $u_j \in M_r$ let $A_j = \Gamma_1(u_j) \cap M_{r-1}$ and $B_j = \bigcup_{w \in A_j} \Gamma_1(w) \cap M_r$, and let $C(M,A_j)$ denote the set of pairs (v,w) such that $v \in M$ and $w \in A_j \cap \Gamma_{r-1}(v)$ (or $v \in \Gamma_{r-1}(w)$). We have $\Gamma_r(u_j) \cap M = \{v \in M \mid \text{there exists a } w \in A_j \text{ such that } (v,w) \in C(M,A_j)\}$. As each $v \in \Gamma_r(u_j) \cap M$ is the first element of at least one pair of $C(M,A_j)$, $|\Gamma_r(u_j) \cap M| \leqslant |C(M,A_j)|$. Since $r \geqslant 2$, $B_j \subseteq \Gamma_r(u_j) \cap M_r$. Therefore, as the r-neighborhood of u_j should contain a majority of vertices of M,

$$|B_j| \leqslant |\Gamma_r(u_j) \cap M| \leqslant |C(M,A_j)|. \tag{7}$$

Now let us select a sequence of vertices from M_r, denoted $u_1, u_2, \ldots, u_p$, as follows. First, pick u_1 to be an arbitrary vertex in M_r. If $B_1 \neq M_r$, then pick u_2 to be an arbitrary vertex in $M_r \setminus B_1$. Repeat this process as long as $\bigcup_{1\leqslant k\leqslant j} B_k \neq M_r$, picking u_{j+1} to be an arbitrary vertex in $M_r \setminus \bigcup_{i\leqslant k\leqslant j} B_k$.

At the end of this process, we have

$$\bigcup_{1\leqslant j\leqslant p} B_j = M_r. \tag{8}$$

By the definition of the B_j's, all the A_j's associated with the chosen u_j are pairwise disjoint. Indeed, $u_j \notin \bigcup_{1\leqslant k\leqslant j-1} B_k$, and so it cannot be adjacent to any vertex of the A_k, $1 \leqslant k \leqslant j-1$.

As the A_j are pairwise disjoint, $C(M,A_i) \cap C(M,A_j) = \emptyset$ for $i \neq j$. So we have

$$\sum_j |C(M,A_j)| \leqslant |C(M,M_{r-1})|. \tag{9}$$

Combining Eq. (8), inequalities (7) and (9) and Fact 2(a), we conclude

$$|M_r| = \left|\bigcup_{1\leqslant j\leqslant p} B_j\right| \leqslant \sum_{1\leqslant j\leqslant r} |B_j| \leqslant \sum_{1\leqslant j\leqslant r} |C(M,A_j)| \leqslant |C(M,M_{r-1})| \leqslant |M|^2.$$

Combined with Fact 2(b), we have that $|M_i| < |M|^2$ for $1 \leqslant i \leqslant r$, and hence $|V \setminus M| \leqslant r|M|^2$; the theorem follows. □

As can be seen from the following theorem, this bound is tight.

Theorem 4.2. *For every fixed* $r \geqslant 2$, $g_r^{\mathrm{SI}}(n) = \mathrm{O}(n^{1/2})$.

Proof. For $r = 2$, we note that in the graph $G^{\mathbf{A}}$ of Fig. 1, the coalition M (presented there as a 1-monopoly) is also a self-ignoring 2-monopoly.

Next, we give an example of such a set for any $r > 2$. For integers r, p, construct the graph $G_{r,p}^{\mathbf{F}}$ as follows. The graph is *leveled*, namely, the vertices are arranged into $\lfloor r/2\rfloor + 2$ levels $1,\ldots,\lfloor r/2\rfloor + 2$, with edges connecting only vertices in adjacent levels ℓ, $\ell + 1$. Level 1 contains p^2 vertices, each level $2 \leqslant \ell \leqslant \lfloor r/2\rfloor + 1$ contains p vertices, and level $\lfloor r/2\rfloor + 2$ contains a single vertex. Let X denote the set of vertices on level 1 and let $M = V \setminus X$. When p is much larger than r, M contains roughly $\sqrt{n}$ vertices, yet the vertices of M majorize all r-neighborhoods of X vertices.

The edges connecting two consecutive levels $\ell - 1$ and ℓ are defined as follows. The single vertex of level $\lfloor r/2\rfloor + 2$ is connected to all the vertices of level $\lfloor r/2\rfloor + 1$. For level $2 \leqslant \ell \leqslant \lfloor r/2\rfloor + 1$, each vertex v is connected to the corresponding vertex at level $\ell - 1$. For level $\ell = 2$, each vertex of level 2 has p distinct neighbors at level 1 (i.e., each vertex of X has exactly one neighbor on level 2). See Fig. 6 for an example graph $G_{r,p}^{\mathbf{F}}$ for $r = 5$, $p = 5$.

A straightforward case analysis reveals that in the graph $G_{r,p}^{\mathbf{F}}$, for every vertex $v \in X$, the majority of the vertices in $\Gamma_r(v)$ are from M. □

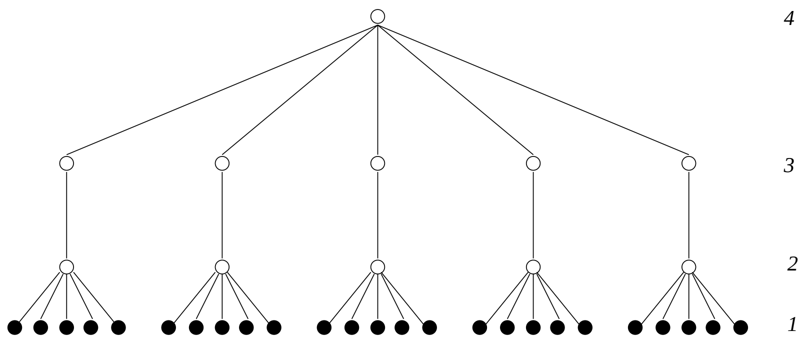

Fig. 6. The graph $G^{\mathbf{F}}_{r,p}$ for $r = 5$ and $p = 5$.

5. [1, r]-controlling coalitions and monopolies

Definition 5.1. Given an n-vertex graph G, a *packing* is a collection $\mathscr{P} = \{P_1, \ldots, P_t\}$ of disjoint neighborhoods in G. For each neighborhood P_i, we denote its center by c_i and its radius by r_i (namely, $P_i = \Gamma_{r_i}(c_i)$ for every i). The *volume* of $\mathscr{P}$ is defined as $\mathscr{V}(\mathscr{P}) = \sum_i |P_i|$.

Definition 5.2. Given a set of vertices X, a packing $\mathscr{P}$ is said to be *X-centered* if all the centers of its neighborhoods are from X.

We make use of the following lemma, which is an extension of Theorem 3.2 of [6] and can be proved along very similar lines. (The proof is omitted.)

Lemma 5.3. *For every n-vertex graph G, set of vertices X and fixed integer r, there exists an X-centered packing $\mathscr{P}$ in G, with neighborhoods of radius at most r, and volume $\mathscr{V}(\mathscr{P}) \geqslant |X|^{1-1/(\lfloor \log_2 r \rfloor + 1)}$. All neighborhoods in the packing may be restricted to have a radius which is a power of* 2.

We now derive a bound on the maximum number of vertices that can be controlled by a coalition M. Let $h_r(m)$ (respectively, $h_{[1,r]}(m)$) denote the maximum number of vertices that can be r-controlled (resp., $[1,r]$-controlled) by a coalition of size m in any graph.

Theorem 5.4. *For every fixed integer $r \geqslant 2$, $h_{[1,r]}(m) = \mathrm{O}(|M|^{1+1/\lfloor \log_2 r \rfloor})$.*

Proof. Consider a coalition M of size m, and let X be the set of vertices that are $[1,r]$-controlled by M. By Lemma 5.3, there exists an X-centered packing $\mathscr{P}$ in G, with neighborhoods of radius at most r, and volume $\mathscr{V}(\mathscr{P}) \geqslant |X|^{1-1/(\lfloor \log_2 r \rfloor + 1)}$. Since the vertices of X are $[1,r]$-controlled by M, each of the neighborhoods in $\mathscr{P}$ contains a majority of vertices from M. By the fact that the neighborhoods in the packing $\mathscr{P}$ are disjoint, $m > \frac{1}{2}|X|^{1-1/(\lfloor\log_2 r\rfloor+1)}$. The claim follows. □

Theorem 5.4 implies that the number of vertices that can be $[1,r]$-controlled by a coalition of size m for $r=2$ or 3 is at most m^2. For $4 \leqslant r \leqslant 7$, that number is bounded by $m^{1.5}$, etc.

We can now prove Theorem 1.12.

Proof of Theorem 1.12. For $r=1$ the claim holds trivially. For $r \geqslant 2$, any self-ignoring $[1,r]$-monopoly M must satisfy $V \setminus M = \mathrm{O}(|M|^{1+1/\lfloor\log_2 r\rfloor})$ by Theorem 5.4, and the claim follows. □

The bounds of Theorems 5.4 and 1.12 are tight. The proof for the existence of a small self-ignoring $[1,r]$-monopoly M is based on a slight modification of the graph $G_{t,p}$ constructed in [6] for establishing the upper bound of Proposition 1.9. The required case analysis is also slightly different. In particular, in [6], neighborhoods of vertices $v \in M$ are considered as well, and therefore majorization is guaranteed only to distance 2^t-1. Hence focusing on the vertices of $V \setminus M$ alone enables majorization in neighborhoods of twice the radius. Also, strict majority is guaranteed on the original graph $G_{t,p}$ in all cases but that of distance 1, in which case a vertex $v \notin M$ has exactly one neighbor in M and one in $V \setminus M$, namely, itself. The construction must therefore be modified to guarantee strict majority in all cases (say, by duplicating each vertex of M which is adjacent to $V \setminus M$, with the same connections). Details are omitted. We have the following.

Theorem 5.5. *For every fixed integer $r \geqslant 1$ there exist (infinitely many) n-vertex graphs G_n and self-ignoring $[1,r]$-monopolies M_n in G_n, such that $|M_n| = \Theta(n^{1-1/(\lfloor\log_2 r\rfloor+1)})$.*

Corollary 5.6. *For every fixed integer $r \geqslant 1$, $g^{\mathrm{SI}}_{[1,r]} = \mathrm{O}(n^{1-1/(\lfloor\log_2 r\rfloor+1)})$.*

As another straightforward corollary, we get that the bound of Theorem 5.4 is tight as well.

Corollary 5.7. *For every fixed integer $r \geqslant 1$ there exist (infinitely many) n-vertex graphs G_n and coalitions M_n, such that M_n $[1,r]$-controls $\Theta(|M_n|^{1+1/\lfloor\log_2 r\rfloor})$ vertices in G_n.*

Acknowledgements

We are grateful to the anonymous referees for their careful reading of the paper and for their helpful comments.

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